

# Monte Carlo Propagation of Spatial Coherence

Scott A. Prahl<sup>a,b</sup>, Donald D. Duncan<sup>b</sup>, David G. Fischer<sup>c</sup>

<sup>a</sup> Oregon Medical Laser Center, Providence St. Vincent Medical Center, Portland, OR

<sup>b</sup> Division of Biomedical Engineering, Oregon Health & Science University, Portland, OR

<sup>c</sup> NASA Glenn Research Center, Cleveland, OH

## ABSTRACT

The propagation of light through complex structures, such as biological tissue, is a poorly understood phenomenon. Current practice typically ignores the coherence of the optical field. Propagation is treated by Monte Carlo implementation of the radiative transport equation, in which the field is taken to be incoherent and is described solely by the first-order statistical moment of the intensity. Although recent Monte Carlo studies have explored the evolution of polarization using a Stokes vector description, these efforts, too are single-point statistical characterizations and thus ignore the wave nature of light. As a result, the manner in which propagation affects coherence and polarization cannot be predicted.

In this paper, we demonstrate a Monte Carlo approach for propagating partially coherent fields through complicated deterministic optical systems. Random sources with arbitrary spatial coherence properties are generated using a Gaussian copula. Physical optics and Monte Carlo predictions of the first and second order statistics of the field are shown for coherent and partially coherent sources for a variety of imaging and non-imaging configurations. Excellent agreement between the physical optics and Monte Carlo predictions is demonstrated in all cases. Finally, we discuss convergence criteria for judging the quality of the Monte Carlo predictions.

Ultimately, this formalism will be utilized to determine certain properties of a given optical system from measurements of the spatial coherence of the field at an output plane. Although our specific interests lie in biomedical imaging applications, it is expected that this work will find application to important radiometric problems as well.

**Keywords:** Partial Coherence, Diffraction, Stochastic, Cross-Spectral Density, Complex Coherence Factor

## 1. INTRODUCTION

Simulating changes in the spatial coherence of an optical field requires two things (1) a method for evolving the field and (2) a method for creating an ensemble of fields with a specified spatial coherence. Traditionally physical optics techniques have been used to evolve the field.<sup>1-3</sup> This leads to challenging integrals that typically need to be calculated by numerically. In this work, Monte Carlo techniques are used to approximate classic Huygen-Fresnel evolution of a field. A copula technique to generate the ensemble of source fields.<sup>4</sup> We compare our results with standard physical optics calculations to show the accuracy and limitation of our method.

This paper is limited to two-dimensional evolution (cylindrical) of light. The field  $U$  is written as

$$U(x, z, t) = U(x, z) e^{i\omega t}$$

and the time dimension is will be omitted for brevity. The axial direction is  $z$  and the transverse direction is  $x$ . The ensemble intensity  $S(x, z)$  and the cross-spectral density  $\mu(x_1, z; x_2, z)$  for a partially coherent field are<sup>1,5</sup>

$$S(x, z) = \langle U(x, z)U^*(x, z) \rangle; \quad \mu(x_1, z; x_2, z) = \frac{\langle U(x_1, z)U^*(x_2, z) \rangle}{\sqrt{S(x_1, z)}\sqrt{S(x_2, z)}} \quad (1)$$

where  $U^*(x, z)$  is the complex conjugate of  $U(x, z)$ . The angle brackets denote ensemble averages over all field realizations. For convenience the (cylindrical) line source is located in the plane  $z = 0$ .

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Send correspondence to [prahl@bme.ogi.edu](mailto:prahl@bme.ogi.edu), OMLC, PSVMC, 9205 SW Barnes Rd., Portland, OR 97225

## 2. MONTE CARLO SIMULATION OF HUYGENS-FRESNEL WAVES

The source field  $U(x, 0)$  is evolved to the plane  $z$  by using a Monte Carlo sampling technique. Each point in the source wavefront is considered to be the origin of a cylindrical wave (Huygens wavelet) emanating from that point. Each wavelet is randomly sampled by launching rays from its center; the initial amplitude and phase of a ray launched from the point  $(x, 0)$  is

$$|U(x, 0)|e^{i\phi(x, 0)}$$

The phase of each ray changes as it moves; a ray that has travelled a distance  $d$  will have  $kd$  added to its phase where  $k$  is the wavenumber ( $k = 2\pi/\lambda$  and  $\lambda$  is the wavelength). When a ray reaches the observation plane, it is added coherently to all others that have reached the same location.

Monte Carlo sampling is simple and flexible, but can be computationally intensive. Typically the relative error decreases as  $\sqrt{N}$  where  $N$  is the number of rays used. Consequently to decrease the error by a factor of ten, the number of rays must be increased by a factor of 100. Variance reduction techniques are often used.

One obvious improvement when using Monte Carlo to sample Huygen's wavelets is to limit the range of angles at which rays are launched to those that will reach the aperture or detector. However, this improvement creates a subtle bias when comparing results from different launch locations. If equal numbers of rays are launched from two locations and the subtended angles differ, then the number of rays/radian will be different. This must be corrected by weighting each ray by the subtended angle used to generate its launch angle. If every source location launches the same number of rays, then the contributions from all source locations will be commensurate and the results can be added together.

Another efficiency improvement is to calculate how a flat coherent source field  $U(x, 0) = 1$  changes as each source element propagates to any detector element. If  $N_s$  is the number of source elements and  $N_d$  is the number of detector elements, then this information can be recorded in an  $N_s \times N_d$  matrix. In the limit of an infinite number of infinitesimal source and detector elements, this mapping is functionally equivalent to a Green's function. For a finite number of elements the detector field can be written in terms of the Green's matrix  $G_{ij}$

$$U_j = G_{ij}U_i$$

where the source field is represented as a vector  $U_i$  and the observation plane fields by  $U_j$ . Monte Carlo is used to generate the values of  $G_{ij}$  by tracing rays from the  $i$ th source element to the  $j$ th detector element.

Young's classic double-slit experiment is a convenient test of our Monte Carlo sampling strategy. The source has constant amplitude across each slit and is perfectly-coherent (so partial coherence is not tested). The source field is unity  $U(x, z) = 1$  within the slits and zero elsewhere. The Fraunhofer approximation for the field intensity resulting from two slits (width  $2a$  separated by  $b$ ) at a distance  $d$  is

$$\frac{S(x, d)}{S(0, d)} = \text{sinc}^2\left(\frac{kax}{\sqrt{d^2 + x^2}}\right) \cos^2\left(\frac{kbx}{2\sqrt{d^2 + x^2}}\right) \quad (2)$$

where  $\text{sinc}(x) = \sin(x)/x$ .

For the simulation shown in Fig. 2, the detector distance ( $10,000\lambda$ ) is larger than  $a^2/\lambda = 100\lambda$  and the Fraunhofer approximation is appropriate. However when the detector elements were  $10\lambda$  wide, enough to adequately sample the  $\sim 120\lambda$  interference peaks in the intensity, the Monte Carlo results were poor. The problem was that the *phase* of the field changes by  $\sim 1$  radian across each detector element and the consequently the intensity fluctuations were averaged out. The solution was to decrease the size of each detector elements to  $1\lambda$ . In this case excellent agreement between Monte Carlo simulations and physical optics calculations was achieved.

The Monte Carlo technique can be used to focus a cylindrical wave by correcting the phase of each ray as it leaves the source. For a focal length of  $f$ , the phase adjustment is  $-ik\sqrt{f^2 + x^2}$  for a ray leaving from a

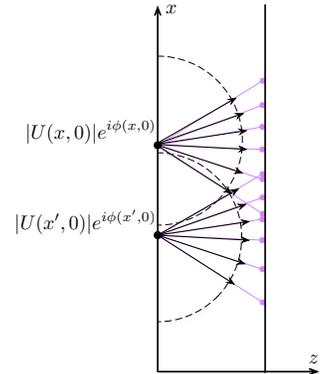


Figure 1. Rays being used to approximate a Huygen's wavelet.

position  $x$ . Fig. 3 shows the intensity in the focal plane of a perfectly-coherent flat field that has passed through a  $f/1$  system. The lateral extent of the source is  $31,620\lambda$  and the focal length is  $31,620\lambda$ . The physical optics calculation<sup>6</sup> for the field at  $x''$  in the focal plane is achieved by integrating over all source positions  $x'$

$$U(x'', f) = \int_{-\infty}^{+\infty} U(x', 0) \frac{i}{\lambda} \exp\left(-ik\sqrt{f^2 + x'^2}\right) \frac{f}{s} H_1^{(1)}(ks) dx'$$

where  $U(x', 0) = 1$  and  $s = \sqrt{(x' - x'')^2 + f^2}$ . This calculation is not amenable to straightforward simplification because the binomial approximation is poor for this geometry.

### 3. GENERATION OF FIELD REALIZATIONS

The Gaussian copula algorithm is used to produce source fields with prescribed amplitude and correlation.<sup>3,4</sup> A Box-Muller transformation is used to produce jointly normal deviates  $Y_1$  and  $Y_2$  from independent uniform deviates  $X_1$  and  $X_2$

$$Y_1 = \sqrt{-2 \ln X_1} \cos(2\pi X_1) \quad \text{and} \quad Y_2 = \sqrt{-2 \ln X_2} \sin(2\pi X_2)$$

These normal deviates are scaled and rotated

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \frac{2\pi}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{1+r} & 0 \\ 0 & \sqrt{1-r} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

to obtain normal deviates  $\phi_1$  and  $\phi_2$  correlated by  $r$ . The phases  $\phi_1$  and  $\phi_2$  are used to fill a matrix with complex numbers having unit amplitude. Spatially band-limited fields are generated by masking this matrix so that it is filled with zeros except for a central circular set of values. Fourier transforming this matrix produces realizations with a correlation given by  $\exp(-\sigma^2/2)$  where  $\sigma^2$  is the variance of the phase difference  $\phi_1 - \phi_2$ . An ensemble of  $2^{16}$  line source realizations were generated with this method. Each line source consists of 51 complex numbers. The ensemble of realizations had constant amplitude and prescribed spatial correlation properties.

Most work involving partial coherence assumes that the fields have Gauss-Schell spatial correlation<sup>1</sup>

$$\langle U^*(x', 0)U(x'', 0) \rangle \equiv \exp\left(-\frac{(x' - x'')^2}{2\sigma_g}\right)$$

where  $x'$  and  $x''$  are points in the source aperture and  $\sigma_g$  is the effective spectral coherence length of the field in the aperture. The ensemble of realizations had constant amplitude and Gauss-Schell spatial correlation. Specifically,  $\sigma_g/a = 0.6$  where  $2a$  was the extent of the source field ( $\sigma_g = 9486\lambda$ ).

The number of fields used in the ensemble affects the fidelity of the ensemble average. In Fig. 4 below, the intensity should be constant and equal to one across the field. When  $2^{16}$  fields were averaged, the intensity remains within a few percent of one across the entire source. As the number of fields that were averaged decreases, the ensemble intensity varies more across the source. For example,  $2^{10}$  fields were needed before the intensity was consistently above 0.9. The spatial coherence tended to converge more rapidly to desired distribution and only required  $2^8$  fields before becoming reasonably accurate. The same sets of realizations used to calculate Fig. 4 were used in the rest of the graphs in this paper.

A primary question about the propagation of partial coherence relates to the convergence properties of the process. The Green's matrix formalism nicely separates the randomness of the source fields from the randomness introduced by the Monte Carlo approximation of the Green's matrix. We make use of this fact in the next two sections to show the sensitivity of the Monte Carlo process on the number of rays used to calculate the Green's matrix and the number of fields propagated with an accurate Green's matrix.

#### 4. THE EFFECT OF THE NUMBER OF RAYS ON PROPAGATION

The influence of the number of rays used to generate the Green's matrix was evaluated first. Different numbers of rays (ranging from  $10^4$  to  $10^9$ ) were used to generate the Green's matrix. For this simple  $f/1$  geometry,  $10^9$  rays took about a minute on a 2 GHz computer. Once the Green's matrix had been estimated, it was used to propagate  $2^{16}$  fields to the focal plane. The ensemble intensity and the cross-spectral density were calculated and plotted along with the more traditional physical optics numerical integration result. The Monte Carlo method obviously works well for large numbers of rays, but what was surprising is that reasonable results could be achieved with only  $10^5$  rays. Since there were 401 detector elements, this works out to an average of 250 rays that reached each bin.

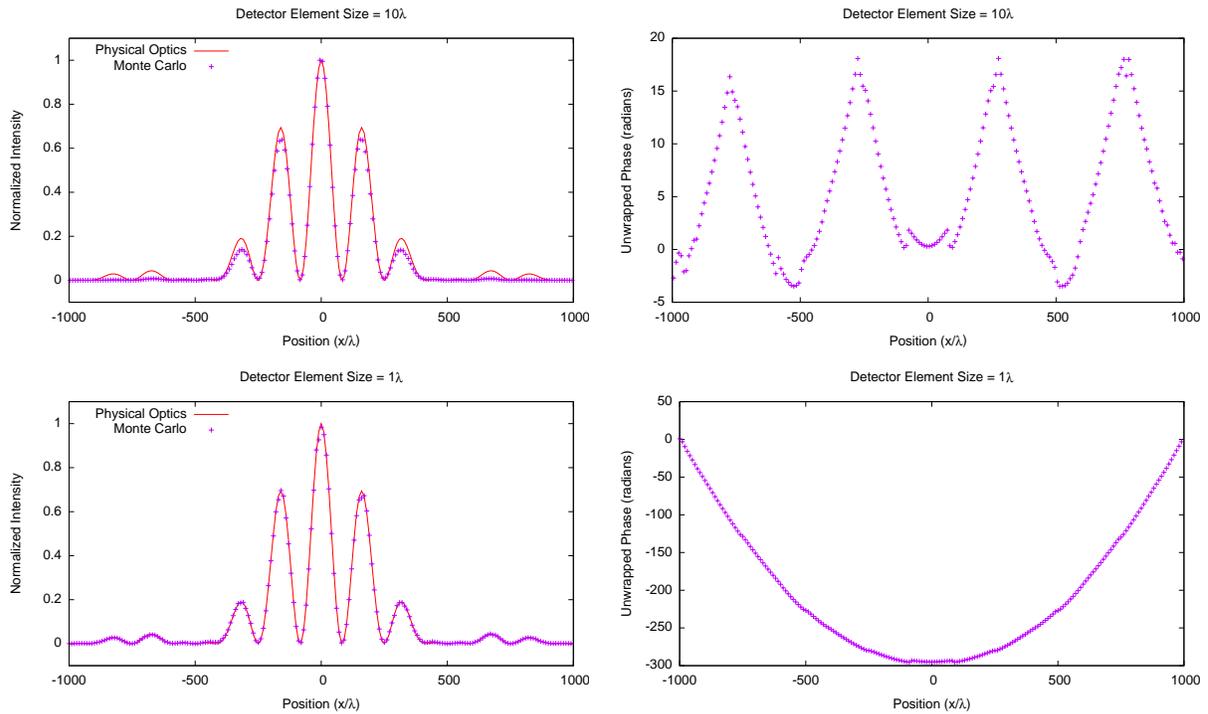


Figure 2. Simulation of Young's two-slit experiment using  $10^8$  rays. The top figures show the intensity and phase when the detector only has 200 elements; the bottom figures show the improvement in results when the number of detector elements is increased to 2000. The slits are  $20\lambda$  wide and have a center-to-center distance of  $60\lambda$ . The detector is at a distance of  $10,000\lambda$ . In the bottom two figures only 200 of the 2000 elements are plotted for clarity.

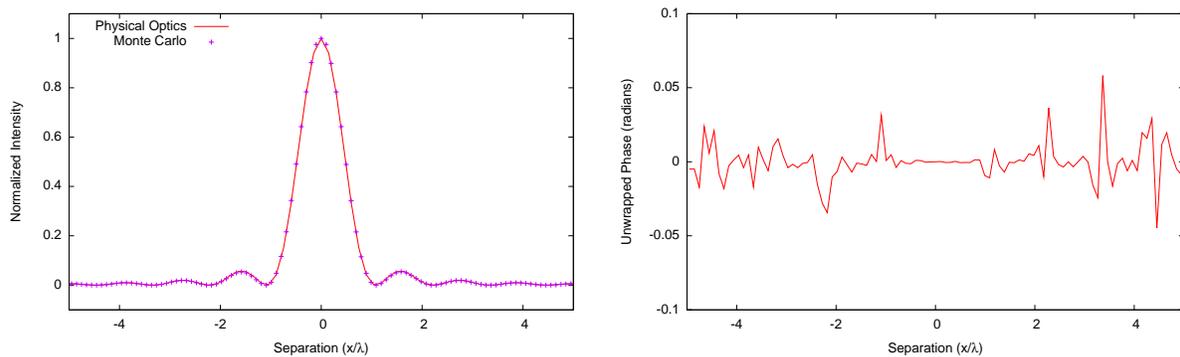


Figure 3. Intensity and phase in the focal plane of a focused perfectly-coherent cylindrical source. This is an  $f/1$  system with  $f = 31,620\lambda$  and  $10^8$  rays were used. Each detector element was  $0.15\lambda$  wide.

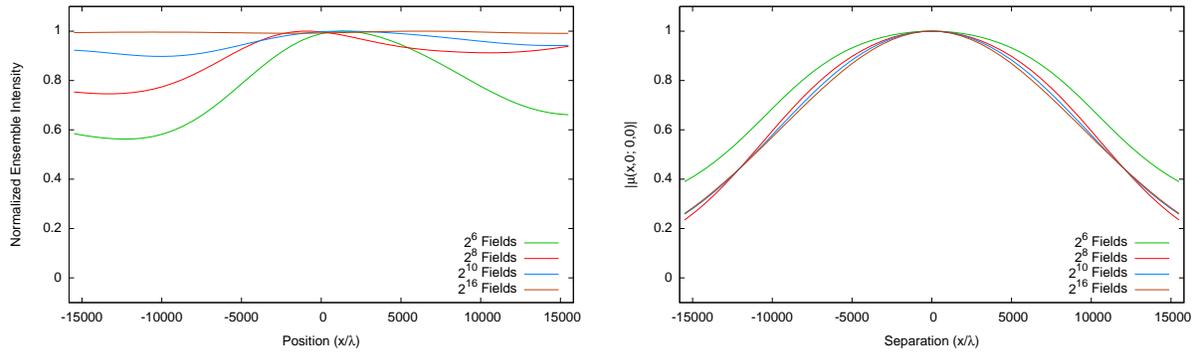


Figure 4. Ensemble averages of the source intensity and the modulus of the cross spectral density  $\mu(x, 0; 0, 0)$ .

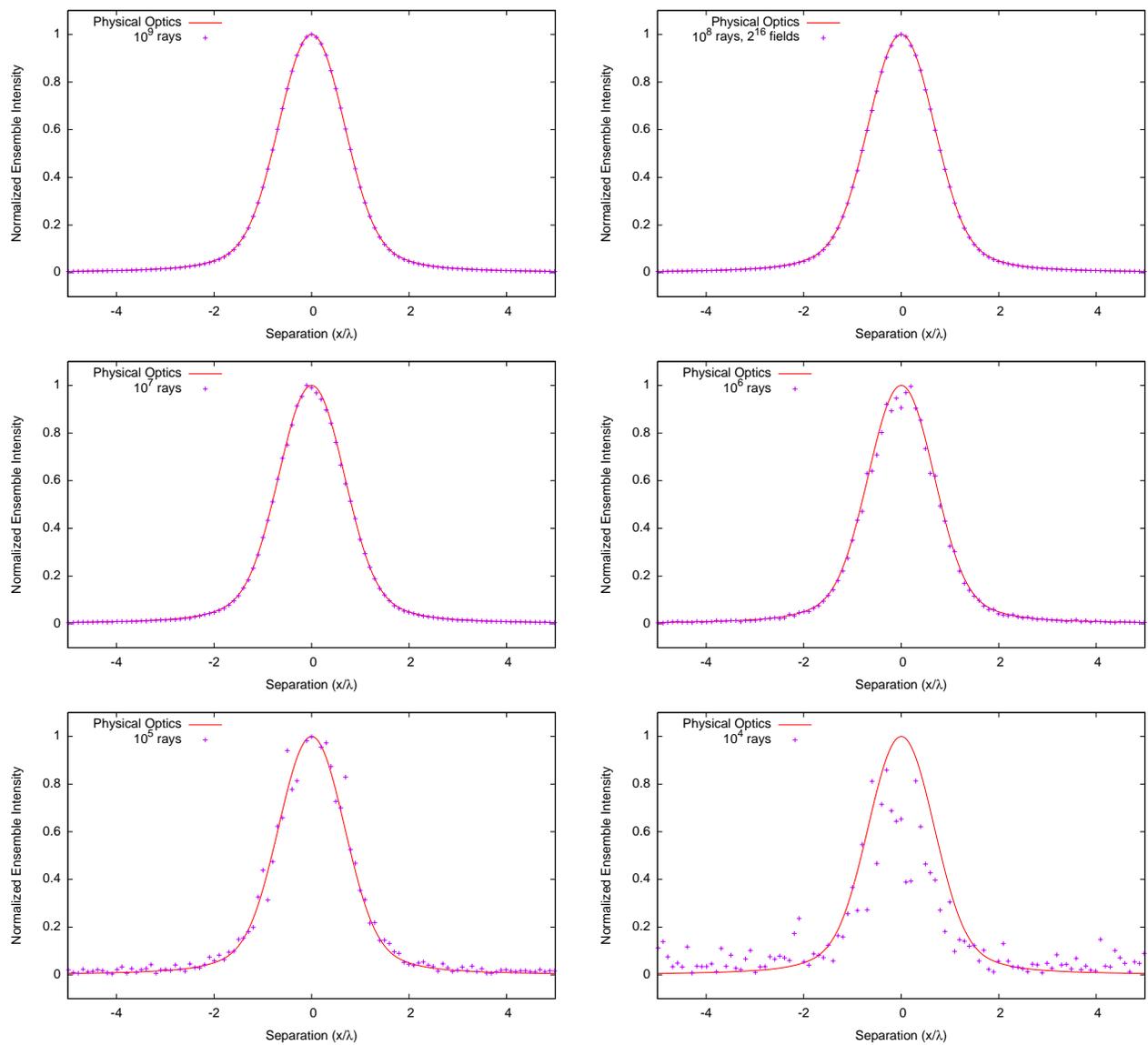


Figure 5. The normalized intensity  $S(x, f)/S(0, f)$  in the focal plane for a focused partially-coherent source. The detector was located in the focal plane at  $z = f = 31,620\lambda$ . Different numbers of rays were used to calculate the Green's matrix  $G_{ij}$  that was used to propagate  $2^{16}$  source fields to the focal plane. The detector had 401 pixels distributed over  $10\lambda$ .

The cross-spectral density was more sensitive to errors in the Green's matrix than the ensemble intensity. As can be seen in the lower left graph of Fig. 6,  $10^6$  rays did a poor job of approximating the cross-spectral density outside the first peak. Increasing the number of rays by a factor of ten to  $10^7$ , improved the approximation significantly.

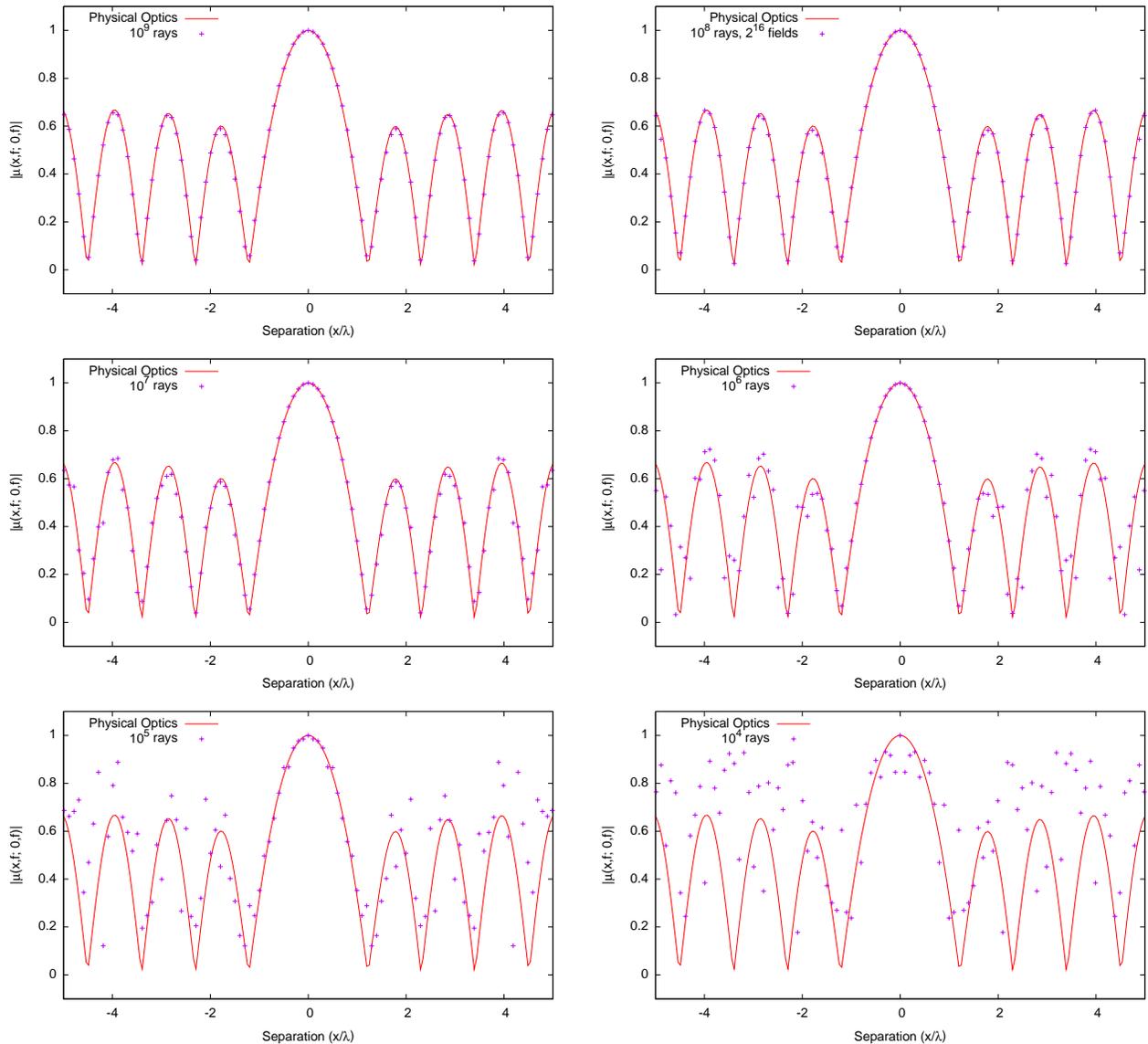


Figure 6. The cross-spectral density  $\mu(x, f; 0, f)$  in the focal plane at  $z = f = 31,620\lambda$ . As indicated in each graph, different numbers of rays were used to calculate the Green's matrix  $G_{ij}$  that was used to propagate  $2^{16}$  source fields to the focal plane. The detector had 401 pixels (only 100 plotted for clarity) and a total lateral extent of  $10.0\lambda$ .

## 5. THE EFFECT OF THE NUMBER OF FIELDS ON PROPAGATION

In this section, the Green's matrix was calculated with  $10^8$  rays using the same  $f/1$  focusing geometry. Different numbers ( $2^6$  to  $2^{16}$ ) of fields were propagated from the source to the focal plane. The ensemble intensity and cross-spectral density were then calculated. The intensities are graphed in Fig. 7 below and show that reasonable accuracy in the ensemble intensity was achieved with only  $2^6$  fields. The cross-spectral density shown in Fig. 8 is also quite good when the same number of fields were used. Considering that Fig. 4 shows a 40% variation in the ensemble source intensity when only  $2^6$  fields were used, these results in the focal plane were surprisingly good.

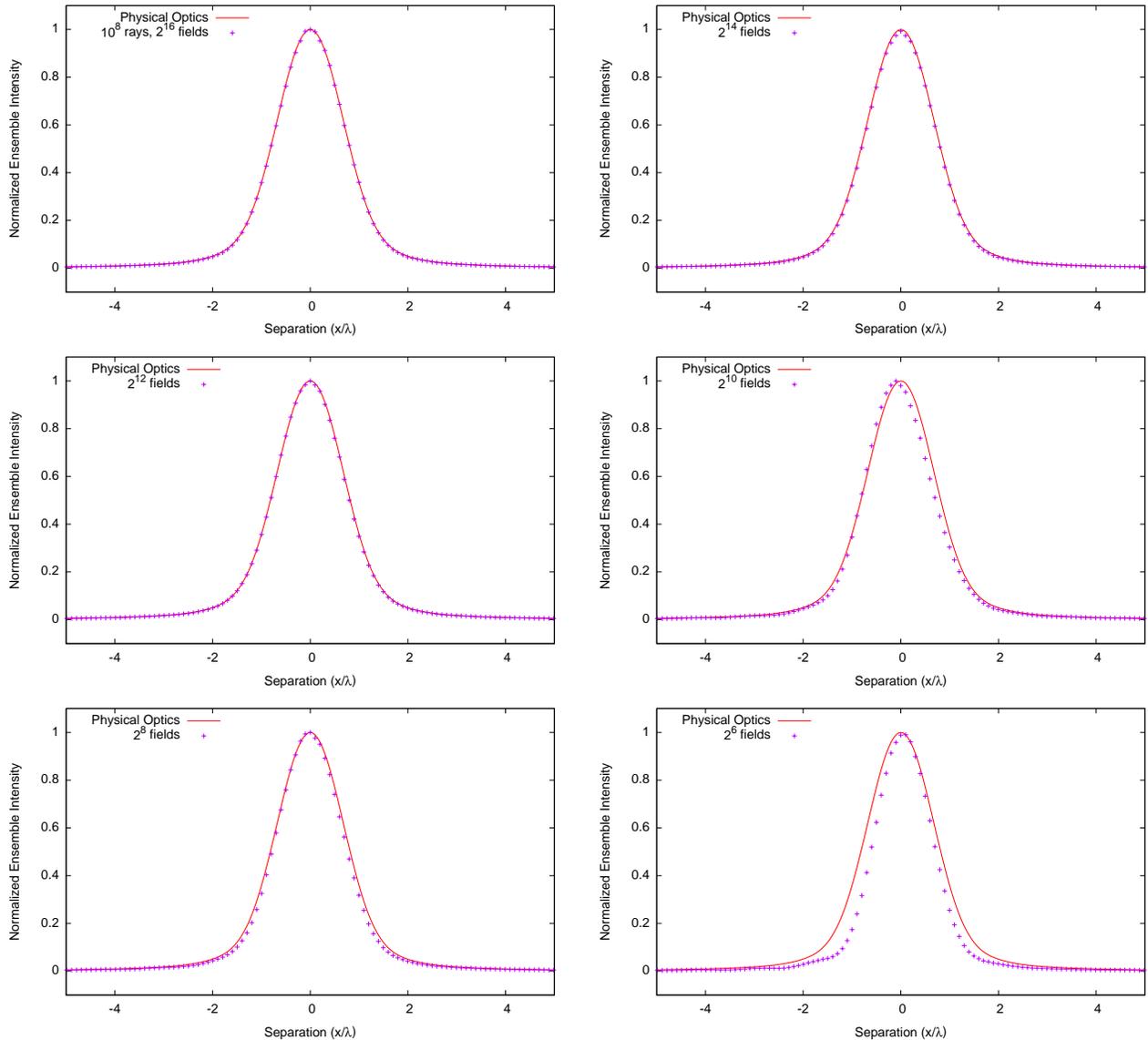


Figure 7. The normalized intensity  $S(x, f)/S(0, f)$  in the focal plane at  $z = f = 31,620\lambda$ . Various numbers of fields were used. The detector had 401 pixels (only 100 plotted for clarity) and is  $10.0\lambda$  wide.

## 6. CONCLUSION

Physical optics and Monte Carlo predictions of the first and second-order statistics of the field have been shown for coherent and partially coherent sources. Excellent agreement between the physical optics and Monte Carlo predictions was demonstrated in all cases. The convergence criteria for judging the quality of the Monte Carlo predictions is shown to depend on the number of rays used as well as on the number of fields propagated. The convergence is more sensitive low numbers of rays than it is to relatively low numbers of fields. The merger of Monte Carlo and the copula method is a simple, yet powerful, tool for studying the propagation of partially coherent fields.

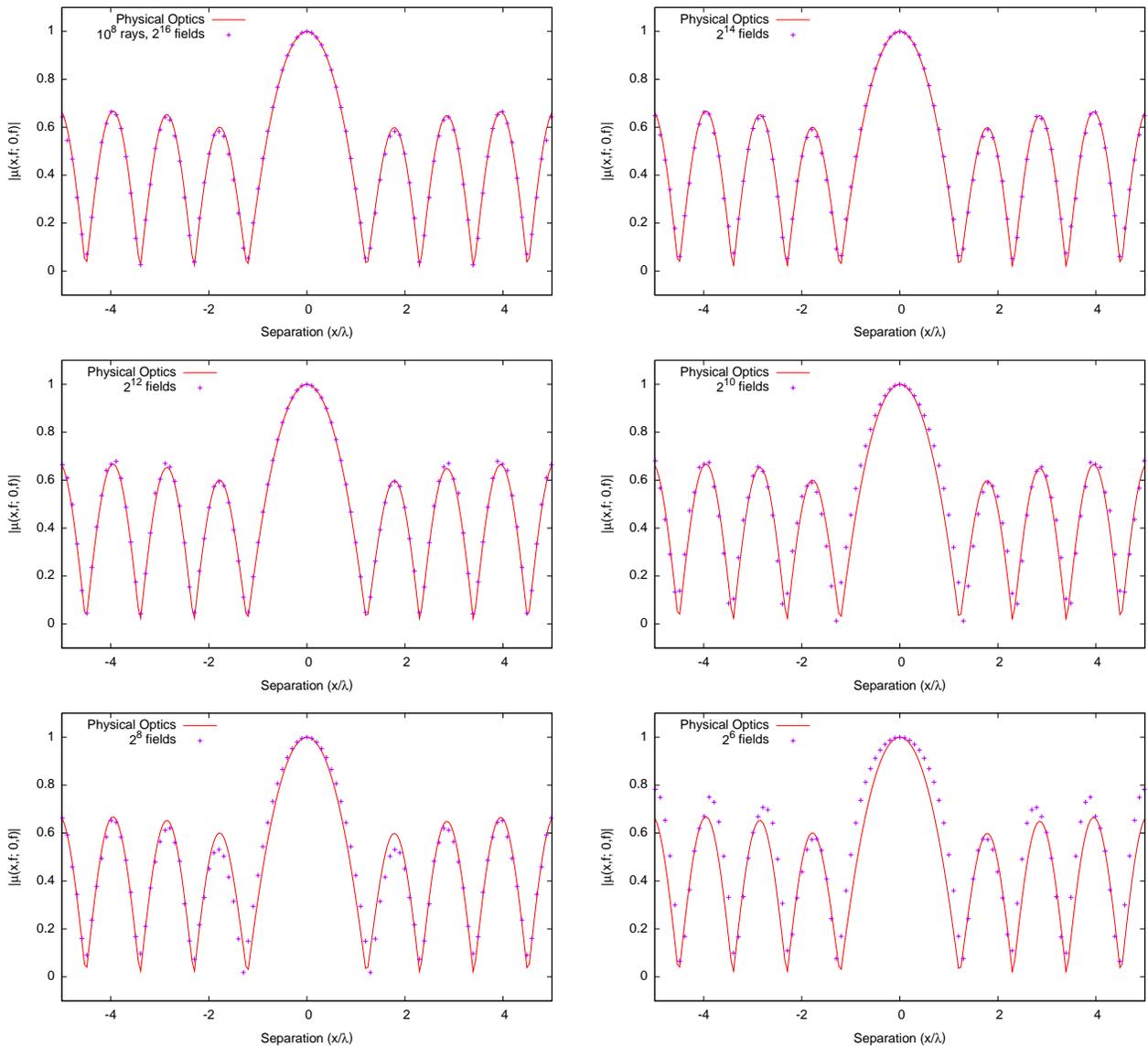


Figure 8. The cross-spectral density  $\mu(x, f; 0, f)$  in the focal plane at  $z = f = 31,620\lambda$ . Various numbers of fields were used. The detector had 401 pixels (only 100 plotted for clarity) and is  $10.0\lambda$  wide.

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