Simple and Accurate Approximations for Reflectance from a Semi-Infinite Turbid Medium

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Abstract: Rational polynomial approximations are given for the total reflection from a semi-infinite turbid medium for normal collimated irradiance. These approximations have an error of less than 0.01 for any albedo or anisotropy.
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OCIS codes: (290.4210) Multiple scattering

A simple reliable method to calculate the total reflection from a semi-infinite scattering and absorbing medium is often convenient. Many accurate radiative transport models are available — discrete ordinates[1], adding-doubling [2, 3], Monte Carlo [4]. These methods are cumbersome to implement when a simple approximation is desired. Alternatively one could use an analytic approximation — diffusion [5] or δ-Eddington approximation [6]. These are much more easily implemented due to their relatively simple forms, but have no guaranteed accuracy.

This paper presents a very simple approximation for the total reflection returning from a semi-infinite medium with normal illumination. The semi-infinite medium is assumed to have a planar boundary and an index of refraction given by n. The index of refraction above the medium is assumed to be unity.

The albedo is $a = \mu_s/(\mu_a + \mu_s)$ where μ_s is the scattering coefficient and μ_a is the absorption coefficient. The single scattering phase function is a Henyey-Greenstein phase function. The average cosine of the phase function is g. Finally, the similarity parameter s is defined as

$$s = \sqrt{\frac{1-a}{1-ag}} = \sqrt{1 - \frac{(1-g)\mu_s}{\mu_a + (1-g)\mu_s}} = \sqrt{1-a'}$$

The benchmark radiative transport calculations needed for this paper made using the adding-doubling approximation with 32 quadrature angles. The phase function was approximated using the δ-M approximation of Wiscombe [7] and mismatched boundaries were approximated using Fresnel reflection as described in Prahl [8]. These calculations were certainly accurate to four places and match those tabulated by van de Hulst [2, 3] to all five digits.

The approximation introduced by van de Hulst [3] for *diffuse* illumination of a semi-infinite medium

$$R_{\text{diffuse illumination}} = \frac{(1-s)(1-0.139s)}{1+1.17s}$$

has an error of less than 0.003 for any albedo and anisotropy. This functional form is convenient because it has the correct limiting behavior for small and large reduced albedos

$$\begin{array}{lcl} R \to 0 & \text{as} & a' \to 0 \\ R \to 1 & \text{as} & a' \to 1. \end{array}$$

This paper adapts van de Hulst's formula to the problem of total reflection, R, for normal collimated illumination and mismatched boundaries

$$R = r_s + (1-r_s)\frac{(1-s)(1-b_1 s)}{1+b_2 s}.$$

Here $r_s = (n-1)^2/(n+1)^2$ accounts for the specular reflection of the collimated illumination. For the simple case of isotropic scattering and matched boundaries $n = 1$, the optimal equation was

$$R = \frac{(1-s)(1-0.128s)}{1+1.83s} \qquad (g = 0, n = 1)$$

Fig. 1. The residual error in the calculation of the total reflection for normal collimated irradiance of a semi-infinite slab with index of refraction $n = 1$ (left) and $n = 1.4$ (right).

This equation has an error of less than 0.002 for any albedo if $g = 0$ and $n = 1$. If this formula is used for other (non-negative) anisotropies, the error may be as large as 0.03.

To reduce the error in the calculation of R to be less than 0.01, it was necessary to find optimal values for b_1 and b_2 for anisotropies ranging from 0 to 1. These values for b_1 and b_2 are in turn approximated by

$$\begin{aligned} b_1 &= -0.13 + 1.20g - 0.63g^2 \qquad (n = 1.0) \\ b_2 &= 1.88 - 1.79g + 1.27g^2 \end{aligned}$$

When b_1 and b_2 are obtained as above, then the error in the reflectance is always less than 0.01 (see Figure 1).

The above process may be repeated for mismatched boundaries. If the index of refraction of the medium is $n = 1.4$ then

$$\begin{aligned} b_1 &= -0.34 - 1.10g + 0.68g^2 \qquad (n = 1.4) \\ b_2 &= 4.11 - 2.77g + 2.27g^2 \end{aligned}$$

These values of b_1 and b_2 also give total reflections with errors less than 0.01 when the index of refraction of the semi-infinite material is 1.4 (Figure 1).

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